ON THE ZETA FUNCTIONS OF PREHOMOGENEOUS VECTOR SPACES FOR PAIR OF SIMPLE ALGEBRAS

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ABSTRACT. In this paper, we consider the prehomogeneous vector space for pair of simple algebras which are k-forms of the D_4 type and the E_6 type. We mainly study the non-split cases. The main purpose of this paper is to determine the principal parts of the global zeta functions associated with these spaces in the non-split cases. We also give a description of the sets of rational orbits of these spaces, which suggests the expected density theorems arising from the properties of these zeta functions.

1. INTRODUCTION

Let k be a field and D a simple algebra of dimension 4 or 9 over k. We denote by D^{op} the opposite algebra of D. In this paper, we consider the prehomogeneous vector space $(G, V) = (G, \rho, V)$ where

$$(1.1) \qquad G = D^\times \times (D^{\mathrm{op}})^\times \times \mathrm{GL}(2), \qquad V = D \otimes k^2,$$

and

$$\rho(g)(a \otimes v) = (g_{11} a g_{12}) \otimes (g_2 v) \quad \text{for} \quad g = (g_{11}, g_{12}, g_2) \in G, a \in D, v \in k^2.$$

We call the prehomogeneous vector space D_4 type and E_6 type if the dimension of D is 4 and 9, respectively. This representation is a k-form of

$$G' = \mathrm{GL}(n) \times \mathrm{GL}(n) \times \mathrm{GL}(2), \qquad V' = k^n \otimes k^n \otimes k^2$$

for $n = 2$ and $n = 3$ if the dimension of D is 4 and 9, respectively. If D is split, then (G, V) is equivariant to (G', V') over k. In this paper, we give a certain description of k-rational orbits and determine the structure of the stabilizers for semi-stable points. Also we determine the principal parts of the global zeta function for the non-split cases of (G, V) over an algebraic number field k.

We recall the definition of prehomogeneous vector spaces. Here we give a definition of a certain restricted class instead of giving general one for simplicity.

Definition 1.1. An irreducible representation of a connected reductive group (G, V) over k is called a *prehomogeneous vector space* if

(1) there exists a Zariski open G-orbit in V and
(2) there exists a non-constant polynomial $P \in k[V]$ and a rational character χ of G such that $P(gx) = \chi(g)P(x)$ for all $g \in G$ and $x \in V$.

Irreducible prehomogeneous vector spaces over an arbitrary characteristic 0 algebraically closed field was classified by Sato and Kimura in [14]. Sato and Shintani

[15] defined global zeta functions for prehomogeneous vector spaces if (G, V) is defined over a number field.

The information of the principal part at the rightmost pole of the global zeta function for a prehomogeneous vector space together with an appropriate local theory yields interesting density theorems. For example, using Shintani's result [16] for the space of binary cubic forms, Datskovsky and Wright [3, 4] gave the zeta function theoretic proof of the Davenport and Heilbronn [5, 6] density theorem

$$\sum_{\substack{[F:\mathbb{Q}]=3 \\ |\Delta_F| \le x}} 1 \sim \frac{x}{\zeta(3)} \qquad (x \to \infty),$$

where F runs through all the cubic fields with the absolute value of its discriminant $|\Delta_F|$ is not bigger than x. Also recent works concerning to the space of pairs of binary Hermitian forms [8, 9, 10] by Kable and Yukie gave certain new density theorems together with Yukie's global theory [21]. For the statement of the density theorem, see the introduction of [8]. Note that this case is another k-form of the D_4 case. These k-forms are listed in H. Saito's classification [12].

We return to our prehomogeneous vector space (1.1). The following theorem is a main result of this paper.

Theorem 1.2. *Let D be a non-split simple algebra of dimension $m = 4$ or 9. The global zeta function $Z(\Phi, s)$ associated with the prehomogeneous vector space (1.1) can be continued meromorphically to the region $\Re(s) > 2m - 2$ only with a possible simple pole at $s = 2m$ with the residue $\tau(G_1)\mathfrak{V}_2 \int_{V_{\mathbb{A}}} \Phi(x)dx$.*

The constants $\tau(G_1), \mathfrak{V}_2$ and the measure dx on $V_{\mathbb{A}}$ are defined in Section 4. All the other poles are also described by means of certain distributions in Theorem 4.24, but the above theorem is enough to get the density theorems. On the other hand, the expected density theorems require not only the standard tauberian theorem, but also the appropriate local theory and what is called "filtering process", which will be studied in a forthcoming paper. For a general transition process from the tauberian theorem for global zeta function of prehomogeneous vector spaces to density theorems, see [20].

The expected density theorems from our cases will be discussed in Remark 3.10 using a result in Section 3, but we also give a brief summary here. We assume $k = \mathbb{Q}$ for simplicity. For a finite extension F of $\mathbb{Q}$, h_F and R_F denote the class number and the regulator of F, respectively.

For a prehomogeneous vector space (G, V) in Definition 1.1, set $V_{\mathbb{Q}}^{\mathrm{ss}} = \{x \in V_{\mathbb{Q}} \mid P(x) \ne 0\}$ where $P(x)$ is as in (2) and $\widetilde{T} = \ker(G \to \mathrm{GL}(V))$. For $x \in V_{\mathbb{Q}}^{\mathrm{ss}}$, G_x denotes the stabilizer of x and G_x° denotes its identity component. Roughly speaking, the global zeta function is a counting function for the unnormalized Tamagawa numbers of $G_x^\circ/\widetilde{T}$ of points in $x \in G_{\mathbb{Q}} \backslash V_{\mathbb{Q}}^{\mathrm{ss}}$ (see [20], for example). Hence, by Proposition 3.6 and 3.7, the behavior of the zeta function at the rightmost pole for the D_4 case and the E_6 case (both split and non-split) yields the asymptotic behavior of the following function

$$\sum_{\substack{[F:\mathbb{Q}]=2 \\ |\Delta_F| \le x \\ F \text{ is embeddable into } D}} h_F^2 R_F^2 \qquad \text{and} \qquad \sum_{\substack{[F:\mathbb{Q}]=3 \\ |\Delta_F| \le x \\ F \text{ is embeddable into } D}} h_F R_F$$

as $x \to \infty$, respectively. If the method of filtering process in [3, 4], [2], [8, 9, 10] is applicable to compute the above density, one can also compute a similar density with

finitely many local conditions by the same method. It is also easy to show that for each D, the condition $F \subset D$ can be determined by finitely many local conditions of F. Hence, the expected density theorems from the non-split cases seems to be quite similar to that of from the split cases.

One advantage of non-split cases is that the global theory becomes much easier. The analysis of the global zeta function becomes much complicated as the split rank of the group growth, and we could not yet success to establish the global theory for the split D_4 and E_6 cases. Especially the rank of the group is 5 for the split E_6 case, and the complexity of computing the principal part of the zeta function seems to be formidable.

For the rest of this section, we will give the contents of this paper and the notations used in this paper. More specific notations will be introduced in each section. In Section 2, we will define the space of pair of simple algebras and summarize its basic properties. Before starting the global theory in Section 4, we will give a certain description of rational orbits in Section 3. The split cases are treated in [19, §3], and this is a slight generalization of those cases. We prove that the set of rational orbits corresponds one to one to a certain set of quadratic extensions and cubic extensions for the D_4 case and the E_6 case, respectively. Also we determine the structure of the stabilizers for semi-stable points. The expected density theorems from our cases will be discussed in Remark 3.10.

In Section 4, we study the global theory for the non-split cases. In §4.1, we introduce notations used in this section and review some basic facts on adelic analysis. In §4.2, we define a global zeta function. Also we will give an estimate of an incomplete theta series. Although H. Saito [13] proved the convergence of the all global zeta functions associated with prehomogeneous vector spaces, we need the estimate in order to use Shintani's lemma. In §4.3, we divide the global zeta function into the "entire part" and the "principal part" by using the Poisson summation formula. We study the "principal part" in later subsections.

In §4.4, we introduce a stratification of unstable points. To separate the contribution from unstable strata, we use Shintani's lemma. In §4.5, we review Shintani's lemma and apply it to our cases. Since $D^\times$ is of rank 0, the smoothed Eisenstein series for our case is essentially the same as that of GL(2). In §4.6, we review some analytic properties of the zeta function associated with (single) simple algebra, because this zeta function appears in the induction process. In §4.7, we compute contributions from unstable points. By putting the results together we have obtained in §4.4–§4.7, we determine the principal part of the global zeta function in §4.8.

The standard symbols $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}$ will denote respectively the rational, real and complex numbers and the rational integers. If R is any ring then $R^\times$ is the set of invertible elements of R and if V is a variety defined over R then V_R denotes its R-points. In Section 2 and 3, k denotes arbitrary field. In Section 4, k denotes an algebraic number field.

2. THE SPACE OF PAIR OF SIMPLE ALGEBRAS

In this section, we define the representation of the space of pair of simple algebras, and discuss its basic properties.

Let k be an arbitrary field and D a simple algebra over k of dimension $m = n^2, n \geq 1$. Let $\mathcal{T}$ and $\mathcal{N}$ be the reduced trace and reduced norm, respectively. We denote by D^{op} the opposite algebra of D. We introduce a group G_1 and its representation space W as follows. Let

$$G_1 = D^\times \times (D^{\mathrm{op}})^\times.$$

That is, G_1 is equal to $D^\times \times D^\times$ set theoretically and the multiplication law is given by $(g_{11}, g_{12})(h_{11}, h_{12}) = (g_{11}h_{11}, h_{12}g_{12})$. We regard G_1 as an algebraic group over k. The simple algebra D can be considered as a vector space over k. When we regard D as a vector space over k, we denote this space as W. We define the action of G_1 on W as follows:

$$(g_1, w) \longmapsto g_{11} w g_{12}, \qquad g_1 = (g_{11}, g_{12}) \in G_1, w \in W.$$

This defines a representation W of G_1. Clearly, (G_1, W) is a prehomogeneous vector space. We discuss the properties of the zeta function associated with this space in §4.6, which will be used in the analysis of the zeta function associated with the space of pair of simple algebras.

Let $G_2 = \mathrm{GL}(2)$ and k^2 the standard representation of G_2. The group $G = G_1 \times G_2$ acts naturally on $V = W \otimes k^2$. This is a k-form of $(\mathrm{GL}(n) \times \mathrm{GL}(n) \times \mathrm{GL}(2), k^n \otimes k^n \otimes k^2)$, and it is proved in [14] that this is a prehomogeneous vector space if and only if $n = 2$ or $n = 3$. Since we are interested in prehomogeneous vector space, *we consider the case $n = 2, 3$ for the rest of this paper*. That is, D is a simple algebra of dimension 4 or 9. We call these representation D_4 type and E_6 type for $n = 2$ and $n = 3$, respectively following [19].

We describe the action more explicitly. Throughout of this paper, we express elements of $V \cong W \oplus W$ as $x = (x_1, x_2)$. We identify $x = (x_1, x_2) \in V$ with $x(v) = v_1 x_1 + v_2 x_2$ which is an element of simple algebra with entries in linear forms in two variables $v = (v_1, v_2)$. Then the action of $g = (g_{11}, g_{12}, g_2) \in G$ on $x \in V$ is defined by

$$(gx)(v) = g_{11} x(v g_2) g_{12}.$$

We put $F_x(v) = \mathcal{N}(x(v))$. This is a binary quadratic form (resp. cubic form) in variables $v = (v_1, v_2)$ if $n = 2$ (resp. $n = 3$), and the discriminant $P(x)$ ($x \in V$) is a polynomial in V. The polynomial $P(x)$ is characterized by

$$P(x) = \prod_{i<j} (\alpha_i \beta_j - \alpha_j \beta_i)^2 \quad \text{for} \quad F_x(v) = \prod_{1 \leq i \leq n} (\alpha_i v_1 - \beta_i v_2), \quad x \in V_{\bar{k}}.$$

Let χ_i ($i = 1, 2$) be the character of G_i defined by

$$\chi_1(g_1) = \mathcal{N}(g_{11})\mathcal{N}(g_{12}), \quad \chi_2(g_2) = \det g_2,$$

respectively. We define $\chi(g) = \chi_1(g_1)^2 \chi_2(g_2)^2$ for $n = 2$ and $\chi(g) = \chi_1(g_1)^4 \chi_2(g_2)^6$ for $n = 3$. Then one can easily see that

$$P(gx) = \chi(g)P(x)$$

and hence $P(x)$ is a relative invariant polynomial with respect to the character χ. Let $S = \{x \in V \mid P(x) = 0\}$ and $V^{\mathrm{ss}} = \{x \in V \mid P(x) \neq 0\}$ and call them the set of unstable points and semi-stable points, respectively. That is, $x \in V$ is semi-stable if and only if $F_x(v)$ does not have a multiple root in $\mathbb{P}^1 = \{(v_1 : v_2)\}$.

3. Rational orbit decomposition

§ 3.1. **Rational orbit decomposition.** In this section, we will interpret the rational orbit space $G_k \backslash V_k^{\mathrm{ss}}$ and determine the structure of the stabilizers for semi-stable points. The split cases are treated in [19, §3], and here is a slight generalization of that. For the expected density theorems, see Remark 3.10. For $x \in V_k^{\mathrm{ss}}$, let G_x be the stabilizer of x and G_x° its identity component.

Let $\mathfrak{Q}_k^{\mathrm{sep}}$ (resp. $\mathfrak{C}_k^{\mathrm{sep}}$) be set of isomorphism classes of separable commutative k-algebras of dimension 2 (resp. 3). For example, $\mathfrak{Q}_k^{\mathrm{sep}}$ can be regarded as the disjoint union of $\{k^2\}$ and the set of separable quadratic extensions of k.

Definition 3.1. For $x \in V_k^{\mathrm{ss}}$, we define
$$Z_x = \operatorname{Proj} k[v_1, v_2]/(F_x(v)),$$
$$\widetilde{k}(x) = \Gamma(Z_x, \mathcal{O}_{Z_x}).$$
Also we define $k(x)$ to be the splitting field of $F_x(v)$.

Note that $\widetilde{k}(x)$ may not be a field. Since V_k^{ss} is the set of x such that F_x does not have a multiple root, Z_x is a reduced scheme over k and $\widetilde{k}(x)$ is an element of $\mathfrak{Q}_k^{\mathrm{sep}}$ (resp. $\mathfrak{C}_k^{\mathrm{sep}}$) for $n = 2$ (resp. $n = 3$). Since
$$F_{gx}(v) = \chi_1(g_1) F_x(v g_2),$$
the isomorphism classes of $Z_x, \widetilde{k}(x)$ and $k(x)$ depend only on the G_k-orbit of x.

We let
$$\widetilde{\alpha}_V : G_k \backslash V_k^{\mathrm{ss}} \longrightarrow \mathfrak{Q}_k^{\mathrm{sep}} \quad (\text{resp. } \mathfrak{C}_k^{\mathrm{sep}}) \quad x \longmapsto \widetilde{k}(x)$$
for the D_4 case (resp. the E_6 case). We first determine the image of $\widetilde{\alpha}_V$.

Definition 3.2. (1) For $n = 2$, we denote by $\mathfrak{Q}_k^{\mathrm{sep}}(D)$ the subset of $\mathfrak{Q}_k^{\mathrm{sep}}$ consisting of algebras which have an embedding to D_k.
(2) For $n = 3$, we denote by $\mathfrak{C}_k^{\mathrm{sep}}(D)$ the subset of $\mathfrak{C}_k^{\mathrm{sep}}$ consisting of algebras which have an embedding to D_k.

Lemma 3.3. (1) *Let (G, V) be the D_4 case. The image of the map $\widetilde{\alpha}_V$ is $\mathfrak{Q}_k^{\mathrm{sep}}(D)$.*
(2) *Let (G, V) be the E_6 case. The image of the map $\widetilde{\alpha}_V$ is $\mathfrak{C}_k^{\mathrm{sep}}(D)$.*
(3) *Moreover, any orbit $G_k x \subset V_k^{\mathrm{ss}}$ contains an element of the form $y = (1, y_2)$.*

Proof. Here we consider the E_6 case. The D_4 case can be treated similarly. First note that for $x = (1, w) \in V_k^{\mathrm{ss}}$, $F_x(v_1, 1) = \mathcal{N}(v_1 + w)$ is the characteristic polynomial of $-w \in D_k$ that does not have a multiple root, and hence the algebra $\widetilde{k}(x)$ is isomorphic to the subalgebra $k[w] \subset D_k$ generated by w over k in D_k.

Let $L \in \mathfrak{C}_k^{\mathrm{sep}}(D)$. We regard L as a subalgebra of D_k and take an element $u \in D_k$ so that $L = k[u]$. Let $x = (1, -u) \in V_k$. Then since $\dim_k L = 3 = \deg F_x(v_1, 1)$, the characteristic polynomial $F_x(v_1, 1)$ of u is also the minimum polynomial of u. Hence $F_x(v_1, 1)$ does not have a multiple root since u is separable. This shows that $x = (1, -u) \in V_k^{\mathrm{ss}}$ and now by the remark above we have $\widetilde{\alpha}_V(G_k x) = L$. This proves that the image of $\widetilde{\alpha}_V$ contains $\mathfrak{C}_k^{\mathrm{sep}}(D)$.

Since (3) implies the opposite inclusion, we consider (3). If D_k is non-split, this is obvious because D_k is a division algebra. We consider the split cases. Since the argument is similar, we consider the E_6 case here. In this case, $D = \mathrm{M}(3)$ be the algebra of 3×3 matrices. For $a \in D$, let $\mathrm{rank}(a)$ denote the rank of the matrix a.

Let $x = (x_1, x_2) \in V_k^{\mathrm{ss}}$. If either the rank of x_1 or x_2 is equal to 3, the element is invertible and hence, there exists a $g \in G_k$ such that $gx = (1, *)$. Also if both the rank of x_1 and x_2 are less than or equal to 1, we have $F_x(v) = \det(x_1 v_1 + x_2 v_2) = 0$ which contradicts to $x \in V_k^{\mathrm{ss}}$. Hence, by changing if necessary, we assume that $\mathrm{rank}(x_1) = 2$. Then there exists a $g_1 \in G_{1k}$ such that $x' = g_1 x = (e, y)$, where

$$e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix}.$$

If $y_{33} = 0$, then it is easy to see that $F_{x'}(v)$ has a multiple root, and so $y_{33} \neq 0$. Hence, again we can take an element $g_1' \in G_{1k}$ so that

$$g_1'(e, y) = (e, z), \quad z = \begin{pmatrix} z_{11} & z_{12} & 0 \\ z_{21} & z_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now it is easy to see that there exist $\alpha, \beta \in k$ so that $\mathrm{rank}(\alpha e + \beta z) = 3$, hence we have (3). $\qquad \square$

We later show that the map $\widetilde{\alpha}_V$ is in fact injective. Next we consider the structure of the stabilizers for semi-stable points. Note that for $x \in V_k^{\mathrm{ss}}$,

$$\dim G_x^\circ = \dim G_x = \dim G - \dim V = 4.$$

Lemma 3.4. *Let $x \in V_k^{\mathrm{ss}}$.*
(1) Let (G, V) be the D_4 case. We have $G_x^\circ \cong (\mathrm{GL}(1)_{\widetilde{k}(x)})^2$ as a group over k.
(2) Let (G, V) be the E_6 case. We have $G_x^\circ \cong \mathrm{GL}(1)_{\widetilde{k}(x)} \times \mathrm{GL}(1)_k$ as a group over k.

Proof. By Lemma 3.3 (3), any G_k-orbit in V_k^{ss} contains an element of the form $x = (1, w)$ with $w \in D_k \setminus k$. Hence it is enough to show the lemma for these elements. We identify $\widetilde{k}(x)$ with $k[w] \subset D_k$.

In order to prove an isomorphism between two algebraic groups G_1, G_2 over k, it is enough to construct isomorphisms between the sets G_{1R}, G_{2R} of R-rational points of G_1, G_2 for all commutative k-algebras R which satisfy the usual functorial property. For this, the reader should see [11, p.17].

We first consider (1). For the D_4 case, $\widetilde{k}(x) = k[w]$ is a separable k-algebra of dimension 2. Let R be any commutative k-algebra. We put $\widetilde{R}(x) = \widetilde{k}(x) \otimes R$. Note that $\widetilde{R}(x) = R[w]$ is a subalgebra of $D_R = D \otimes R$ and is commutative. Since $\{1, w\}$ is a k-basis of $\widetilde{k}(x)$, this is also an R-basis of $\widetilde{R}(x)$. Let $s, t \in \widetilde{R}(x)^\times$. Then $\{st, stw\}$ is also an R-basis of $\widetilde{R}(x)$, and so there exists a unique element $g = g_{st} \in \mathrm{GL}(2)_R$ such that $g\,{}^t(st, stw) = {}^t(1, w)$. Hence $(s, t) \mapsto (s, t, g_{st})$ gives an injective homomorphism from $(\widetilde{R}(x)^\times)^2$ to G_{xR}.

This shows that there exists an injective homomorphism

$$(\mathrm{GL}(1)_{\widetilde{k}(x)})^2 \longrightarrow G_x.$$

Since $(\mathrm{GL}(1)_{\widetilde{k}(x)})^2$ is a connected algebraic group of dimension 4, we have $(\mathrm{GL}(1)_{\widetilde{k}(x)})^2 \cong G_x^\circ$.

Next we consider (2). Again we let R be any algebra and put $\widetilde{R}(x) = \widetilde{k}(x) \otimes R$. Then we have a injective homomorphism from $\widetilde{R}(x)^\times \times R^\times$ to G_{xR} by sending (s, t) to

$(s, s^{-1}t^{-1}, t)$. This shows that there exists an injective homomorphism

$$\mathrm{GL}(1)_{\widetilde{k}(x)} \times \mathrm{GL}(1)_k \longrightarrow G_x.$$

Since $\mathrm{GL}(1)_{\widetilde{k}(x)} \times \mathrm{GL}(1)_k$ is a connected algebraic group of dimension 4, we have $\mathrm{GL}(1)_{\widetilde{k}(x)} \times \mathrm{GL}(1)_k \cong G_x^\circ$. $\qquad\square$

Finally, we show the injectivity of $\widetilde{\alpha}_V$.

Lemma 3.5. *In both cases, the map $\widetilde{\alpha}_V$ is injective.*

Proof. Since the split case is already proven in [19], we only consider the non-split cases here. Let $x, y \in V_k^{\mathrm{ss}}$ satisfy $\widetilde{k}(x) \cong \widetilde{k}(y)$. By Lemma 3.3 (3), we may assume $x = (1, u_1), y = (1, u_2)$. Then $k[u_1]$ and $k[u_2]$ are isomorphic subfields of D_k. By the Skolem-Noether theorem [1, CHAPITRE 8 §10], there exists an element $\theta \in D_k^\times$ such that

$$k[u_1] \longrightarrow k[u_2], \quad p \longmapsto \theta p \theta^{-1}$$

gives an isomorphism.

Let (G, V) be the D_4 case. Then $k[u_1]$ is a quadratic extension over k. Hence there exist $a, b \in k$ with $b \neq 0$ such that $u_2 = \theta(a + bu_1)\theta^{-1}$. Hence for

$$g = \left(\theta, \theta^{-1}, \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} \right) \in G_k,$$

we have $y = gx$.

Let (G, V) be the E_6 case. There exists $p \in k[u_1]$ so that $u_2 = \theta p \theta^{-1}$. We claim that there exist $a, b, c, d \in k$ with $ad - bc \neq 0$ such that $p = (c + du_1)/(a + bu_1)$. In fact, if we consider the k-linear map

$$\psi \colon k^4 \longrightarrow k[u_1], \qquad (a, b, c, d) \longmapsto (a + bu_1)p - (c + du_1),$$

the kernel of ψ is non-trivial. Therefore there exists $(a, b, c, d) \in k^4 \setminus \{0\}$ so that $(a + bu_1)p - (c + du_1) = 0$, and since $p \notin k$, we have $ad - bc \neq 0$.

Hence for

$$g = \left(\theta(a + bu_1)^{-1}, \theta^{-1}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \in G_k,$$

we have $y = gx$. $\qquad\square$

We summarize the result in this subsection as follows.

Proposition 3.6. *Let (G, V) be the D_4 case.*
(1) *The map*

$$G_k \backslash V_k^{\mathrm{ss}} \longrightarrow \mathfrak{Q}_k^{\mathrm{sep}}(D), \quad x \longmapsto \widetilde{k}(x)$$

is bijective.
(2) *Let $x \in V_k^{\mathrm{ss}}$. As a group over k, $G_x^\circ \cong (\mathrm{GL}(1)_{\widetilde{k}(x)})^2$.*

Proposition 3.7. *Let (G, V) be the E_6 case.*
(1) *The map*

$$G_k \backslash V_k^{\mathrm{ss}} \longrightarrow \mathfrak{C}_k^{\mathrm{sep}}(D), \quad x \longmapsto \widetilde{k}(x)$$

is bijective.
(2) *Let $x \in V_k^{\mathrm{ss}}$. As a group over k, $G_x^\circ \cong \mathrm{GL}(1)_{\widetilde{k}(x)} \times \mathrm{GL}(1)_k$.*

§ 3.2. **Application to global fields.** If k is a global field, it is well known that the sets $\mathfrak{Q}_k^{\mathrm{sep}}(D), \mathfrak{C}_k^{\mathrm{sep}}(D)$ can be described by means of local conditions. Here, we review the argument. We assume that k is a global field in this subsection. Also if D is split, $\mathfrak{Q}_k^{\mathrm{sep}}(D) = \mathfrak{Q}_k^{\mathrm{sep}}$ and $\mathfrak{C}_k^{\mathrm{sep}}(D) = \mathfrak{C}_k^{\mathrm{sep}}$ for $n = 2$ and $n = 3$, respectively. Hence we assume D is non-split in this subsection. Recall that $m = n^2$ is the dimension of D.

Let $\mathfrak{M}$ be the set of places of k. For $v \in \mathfrak{M}$, let k_v be the completion of k at v. We denote by $\mathrm{Inv}_v(D)$ the Hasse invariant of $D \otimes k_v$ over k_v.

Definition 3.8. We define $\mathfrak{M}_D$ to be the set of elements $v \in \mathfrak{M}$ which satisfy $\mathrm{Inv}_v(D) \neq 0$.

It is well known that $\mathfrak{M}_D$ is a finite set.

Proposition 3.9. (1) *For $n = 2$, the set $\mathfrak{Q}_k^{\mathrm{sep}}(D)$ consists of elements $L \in \mathfrak{Q}_k^{\mathrm{sep}}$ such that $L \otimes k_v$ is a quadratic extension of k_v for all $v \in \mathfrak{M}_D$.*
(2) *For $n = 3$, the set $\mathfrak{C}_k^{\mathrm{sep}}(D)$ consists of elements $L \in \mathfrak{C}_k^{\mathrm{sep}}$ such that $L \otimes k_v$ is a cubic extension of k_v for all $v \in \mathfrak{M}_D$.*

Proof. We will prove the case $n = 3$. The case $n = 2$ can be treated similarly.

Let L be an arbitrary separable cubic extension of k. We denote by $\mathfrak{M}_L$ the set of places of L. The field L is an element of $\mathfrak{C}_k^{\mathrm{sep}}(D)$ if and only if D is split over L. By the Hasse principle, this condition is equivalent to that $D \otimes_k L_w \cong \mathrm{M}(3,3)_{L_w}$ for all $w \in \mathfrak{M}_L$. Since $D \otimes_k k_v \cong \mathrm{M}(3,3)_{k_v}$ for all $v \notin \mathfrak{M}_D$, we only need to consider w which divides an element $v \in \mathfrak{M}_D$. For this v, $D_v = D \otimes k_v$ is a division algebra. Hence for a separable extension F/k_v with $[F : k_v] \leq 3$, $D_v \otimes_{k_v} F \cong \mathrm{M}(3,3)_F$ if and only if $[F : k_v] = 3$. Therefore $D_v \otimes_{k_v} L_w \cong \mathrm{M}(3,3)_{L_w}$ if and only if $[L_w : k_v] = 3$. $\qquad\square$

Remark 3.10. Let $\widetilde{T} = \ker(G \to \mathrm{GL}(V))$. Then it is easy to that

$$\widetilde{T} = \{(t_{11}, t_{12}, t_2) \mid t_{11}, t_{12}, t_2 \in \mathrm{GL}(1)_k, t_{11}t_{12}t_2 = 1\} \cong \mathrm{GL}(1)_k \times \mathrm{GL}(1)_k,$$

and hence

$$G_x^\circ/\widetilde{T} \cong \begin{cases} (\mathrm{GL}(1)_{\widetilde{k}(x)}/\mathrm{GL}(1)_k)^2 & \text{the } D_4 \text{ case,} \\ \mathrm{GL}(1)_{\widetilde{k}(x)}/\mathrm{GL}(1)_k & \text{the } E_6 \text{ case.} \end{cases}$$

For the non-split cases, $\widetilde{k}(x)$ is a quadratic or cubic field over k for any $x \in V_k^{\mathrm{ss}}$, and hence $G_x^\circ/\widetilde{T}$ does not contain a split torus. This shows that (G, V) are of complete type for the non-split cases.

We conclude this subsection with a brief discussion of the possible density theorems which we expect to derive from the theory of the zeta function for our cases.

Roughly speaking, the global zeta function is a counting function for the unnormalized Tamagawa numbers of $G_x^\circ/\widetilde{T}$ of points in $x \in G_k \backslash V_k^{\mathrm{ss}}$. Let $F = \widetilde{k}(x)$ and we denote by h_F and R_F the class number and the regulator of F, respectively. If we consider the canonical measure on the adelization of $G_x^\circ/\widetilde{T}$, the unnormalized Tamagawa number of this group is $(\mathrm{Res}_{s=1}\zeta_F(s))^2$ (resp. $\mathrm{Res}_{s=1}\zeta_F(s)$) for the D_4 case (resp. the E_6 case) where $\zeta_F(s)$ is the Dedekind zeta function. This leads us to believe that the theory of the zeta function will eventually yield the average density of $h_F^2 R_F^2$ for $F \in \mathfrak{Q}_k^{\mathrm{sep}}(D)$ from the D_4 case, and the average density of $h_F R_F$ for $F \in \mathfrak{C}_k^{\mathrm{sep}}(D)$ from the E_6 case.

So, we have to develop a local theory to see that these are the correct interpretation of the problem.

4. The global zeta function

In this section, we study analytic properties of the global zeta function for non-split cases. The main result is Theorem 4.24, which describe the principal parts of the global zeta function.

§ 4.1. **Preliminaries.** In this subsection, we collect basic notations that we use in this section. Also, we review some basic facts concerning adelic analysis that we need later. Throughout this section, k is a number field. Let D be a non-split simple algebra of k of dimension 4 or 9. Then D is a division algebra. Since the argument is similar for the two cases, we treat them simultaneously. Recall that $m = n^2$ is the dimension of D.

Suppose that G is a locally compact group and Γ a discrete subgroup of G contained in the maximal unimodular subgroup of G. For any left invariant measure dg on G, we choose a left invariant measure dg (we use the same notation, but the meaning will be clear from the context) on $X = G/\Gamma$ so that

$$\int_G f(g)\, dg = \int_X \sum_{\gamma \in \Gamma} f(g\gamma)\, dg.$$

Let r_1, r_2, h_k, R_k and Δ_k be the number of real places, the number of complex places, the class number, the regulator and the discriminant of k, respectively. Let e_k be the number of roots of unity contained in k. We set

$$\mathfrak{c}_k = 2^{r_1}(2\pi)^{r_2} h_k R_k e_k^{-1}.$$

We refer [17] as the basic reference for fundamental properties on adeles. The ring of adeles and the group of ideles are denoted by $\mathbb{A}$ and $\mathbb{A}^\times$, respectively. The adelic absolute value $|\ |$ on $\mathbb{A}^\times$ is normalized so that, for $t \in \mathbb{A}^\times$, $|t|$ is the module of multiplication by t with respect to any Haar measure dx on $\mathbb{A}$, i.e. $|t| = d(tx)/dx$. Let $\mathbb{A}^0 = \{t \in \mathbb{A}^\times \mid |t| = 1\}$. We fix a non-trivial additive character $\langle\ \rangle$ of $\mathbb{A}/k$. The set of positive real numbers is denoted $\mathbb{R}_+$. Suppose $[k : \mathbb{Q}] = n$. For $\lambda \in \mathbb{R}_+$, $\underline{\lambda} \in \mathbb{A}^\times$ is the idele whose component at any infinite place is $\lambda^{1/n}$ and whose component at any finite place is 1. Then we have $|\underline{\lambda}| = \lambda$.

We choose a Haar measure dx on $\mathbb{A}$ so that $\int_{\mathbb{A}/k} dx = 1$. We define a Haar measure $d^\times t^0$ on $\mathbb{A}^0$ so that $\int_{\mathbb{A}^0/k^\times} d^\times t^0 = 1$. Using this measure, we choose a Haar measure $d^\times t$ on $\mathbb{A}^\times$ so that

$$\int_{\mathbb{A}^\times} f(t)\, d^\times t = \int_0^\infty \int_{\mathbb{A}^0} f(\underline{\lambda} t^0)\, d^\times \lambda d^\times t^0,$$

where $d^\times \lambda = \lambda^{-1} d\lambda$ and $d\lambda$ is a Lebesgue measure.

Let $\zeta_k(s)$ be the Dedekind zeta function of k. We define

$$Z_k(s) = |\Delta_k|^{s/2} \left(\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\right)^{r_1} \left((2\pi)^{1-s}\Gamma(s)\right)^{r_2} \zeta_k(s).$$

This definition differs from that in [17], p.129 by the factor of $|\Delta_k|^{s/2}$ and from that in [20] by the factor of $(2\pi)^{r_2}$. It is adopted here as the most convenient for our purposes. It is well known that $\operatorname{Res}_{s=1} Z_k(s) = \mathfrak{c}_k$. We define

$$\phi(s) = \frac{Z_k(s)}{Z_k(s+1)}, \quad \varrho = \operatorname{Res}_{s=1}\phi(s) = \frac{\mathfrak{c}_k}{Z_k(2)}, \quad \text{and} \quad \mathfrak{V}_2 = \varrho^{-1}.$$

For a complex variable s, we denote by $\Re(s)$ the real part.

For a vector space V over k, $V_{\mathbb{A}}$ denotes its adelization. Let $\mathcal{S}(V_{\mathbb{A}})$ be the spaces of Schwartz-Bruhat functions on $V_{\mathbb{A}}$. We define the Haar measure dx on $V_{\mathbb{A}}$ so that $\int_{V_{\mathbb{A}}/V_k} dx = 1$.

We express elements of $G_2 = \mathrm{GL}(2)$ as follows:

$$a(t_1, t_2) = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}, \ n(u) = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}, \ \nu = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We recall the following well known facts concerning adelic analysis. The proof may be found in [20, Chapter 1].

Lemma 4.1. (1) *Let $C \subset \mathrm{GL}(V)_{\mathbb{A}}$ be a compact set, and $\Phi \in \mathcal{S}(V_{\mathbb{A}})$. Then there exists $\Psi \in \mathcal{S}(V_{\mathbb{A}})$ such that*

$$|\Phi(gx)| \leq \Psi(x)$$

for all $g \in C, x \in V_{\mathbb{A}}$.

(2) *Let Φ be a Schwartz-Bruhat function on $\mathbb{A}^n$. Then there exist Schwartz-Bruhat functions $\Phi_1, \ldots, \Phi_n \geq 0$ such that*

$$|\Phi(x_1, \ldots, x_n)| \leq \Phi_1(x_1) \cdots \Phi_n(x_n).$$

(3) *Suppose $\Phi \in \mathcal{S}(\mathbb{A})$. Then for any $N \geq 1$,*

$$\sum_{x \in k} \Phi(tx) \ll \max\{1, |t|^{-1}\}, \qquad \sum_{x \in k^{\times}} \Phi(tx) \ll |t|^{-N}.$$

§ 4.2. **The global zeta function.** In this subsection, we define the global zeta function. Also we give an estimate of an incomplete theta series in order to use Shintani's lemma.

Recall that $G_2 = \mathrm{GL}(2)$. Let $T_2 \subset G_2$ be the set of diagonal matrices and $N_2 \subset G_2$ be the set of lower-triangular matrices whose diagonal entries are 1. Then $B_2 = T_2 N_2$ is a Borel subgroup of G_2.

Let

$$G_{1\mathbb{A}}^0 = \{g_1 = (g_{11}, g_{12}) \in G_{1\mathbb{A}} \mid |\mathcal{N}(g_{11})| = |\mathcal{N}(g_{12})| = 1\},$$
$$G_{2\mathbb{A}}^0 = \{g_2 \in G_{2\mathbb{A}} \mid |\det g_2| = 1\},$$
$$G_{\mathbb{A}}^0 = G_{1\mathbb{A}}^0 \times G_{2\mathbb{A}}^0, \quad \overline{G}_{\mathbb{A}} = \mathbb{R}_+ \times G_{\mathbb{A}}^0,$$
$$\widehat{T}_{2\mathbb{A}}^0 = \{a(t_{21}, t_{22}) \mid t_{21}, t_{22} \in \mathbb{A}^0\},$$
$$T_{2\mathbb{A}}^0 = \{a(\underline{\mu}^{-1}, \underline{\mu})t_2 \mid \mu \in \mathbb{R}_+, t_2 \in \widehat{T}_{2\mathbb{A}}^0\},$$
$$B_{2\mathbb{A}}^0 = T_{2\mathbb{A}}^0 N_{2\mathbb{A}}, \quad P_{\mathbb{A}}^0 = G_{1\mathbb{A}}^0 \times B_{2\mathbb{A}}^0.$$

The group $\overline{G}_{\mathbb{A}}$ acts on $V_{\mathbb{A}}$ by assuming that $\lambda \in \mathbb{R}_+$ acts by multiplication by $\underline{\lambda}$. Throughout this section, we express elements $\bar{g} \in \overline{G}_{\mathbb{A}}, g^0 \in G_{\mathbb{A}}^0$ as

$$\bar{g} = (\lambda, g_1, g_2), g^0 = (g_1, g_2)$$

where $\lambda \in \mathbb{R}_+, g_1 \in G_{1\mathbb{A}}^0$, and $g_2 \in G_{2\mathbb{A}}^0$. We identify element $g^0 \in G_{\mathbb{A}}^0$ with $(1, g^0) \in \overline{G}_{\mathbb{A}}$ and $g_1 \in G_{1\mathbb{A}}^0, g_2 \in G_{2\mathbb{A}}^0$ with $(1, g_1, 1), (1, 1, g_2)$. We may also write as $\bar{g} = \underline{\lambda}g^0$.

Let K_2 be the standard maximal compact subgroup of $G_{2\mathbb{A}}^0$ i.e.

$$K_2 = \prod_{v \in \mathfrak{M}_{\mathbb{R}}} \mathrm{O}(2, \mathbb{R}) \times \prod_{v \in \mathfrak{M}_{\mathbb{C}}} \mathrm{U}(2, \mathbb{C}) \times \prod_{v \in \mathfrak{M}_{\mathrm{f}}} \mathrm{GL}(2, \mathcal{O}_v)$$

Let $d\kappa_2$ be the Haar measure on K_2 such that the total volume of K_2 is 1.

Let
$$t_2 = a_2(\underline{\mu}^{-1}t_{21}, \underline{\mu}t_{22}), b_2 = t_2 n_2(u)$$
where $\mu \in \mathbb{R}_+, t_{21}, t_{22} \in \mathbb{A}^0, u \in \mathbb{A}$. Throughout this section, we assume that
$$g_2 = \kappa_2 b_2 = \kappa_2 a_2(\underline{\mu}^{-1}t_{21}, \underline{\mu}t_{22})n_2(u)$$
is the Iwasawa decomposition of $g_2 \in G_{2\mathbb{A}}^0$.

The measure du on $\mathbb{A}$ induces an invariant measure on $N_\mathbb{A}$. We put
$$d^\times t = d^\times \mu \, d^\times t_{21} \, d^\times t_{22}, db_2 = \mu^2 \, d^\times t_2 \, du_2$$
We use $dg_2 = d\kappa_2 db_2^0$ as the Haar measure on $G_{2\mathbb{A}}^0$. It is well known that the volume of $G_{2\mathbb{A}}^0/G_{2k}$ with respect to the measure dg_2 is $\mathfrak{V}_2$.

We fix an arbitrary Haar measure dg_1 on $G_{1\mathbb{A}}^0$. Since the rank of the group G_1 is 0, $G_{1\mathbb{A}}^0/G_{1k}$ is compact. We put
$$\tau(G_1) = \int_{G_{1\mathbb{A}}^0/G_{1k}} dg_1.$$
We choose $dg^0 = dg_1 dg_2, d\bar{g} = d^\times \lambda dg^0$ as Haar measures on $G_\mathbb{A}^0, \overline{G}_\mathbb{A}$, respectively.

For $\eta > 0$, we define
$$T_{2\eta+}^0 = \{a(\underline{\mu}^{-1}, \underline{\mu}) \mid \mu \in \mathbb{R}_+, \mu \le \eta\}.$$
Let $\Omega_2 \subset \widehat{T}_{2\mathbb{A}}^0 N_{2\mathbb{A}}$ be a compact subset. We define $\mathfrak{S}_2^0 = K_2 T_{2\eta+}^0 \Omega_2$. It is well known that for a suitable choice of η and Ω_2, $\mathfrak{S}_2^0$ surjects to $G_{2\mathbb{A}}^0/G_{2k}$. Also there exists another compact set $\widehat{\Omega}_2 \in G_{2\mathbb{A}}^0$ such that $\mathfrak{S}_2^0 \subset \widehat{\Omega}_2 T_{2\eta+}^0$. We fix a compact subset $\widehat{\Omega}_1 \subset G_{1\mathbb{A}}^0$ which surjects to $G_{1\mathbb{A}}^0/G_{1k}$. Let $\widehat{\Omega} = \widehat{\Omega}_1 \times \widehat{\Omega}_2$.

Definition 4.2. Let $r \in \mathbb{R}$. We define $C(G_\mathbb{A}^0/G_k, r)$ to be the set of continuous functions $f(g^0)$ on $G_\mathbb{A}^0/G_k$ satisfying
$$\sup_{g^0 \in \widehat{\Omega} T_{2\eta+}^0} f(g^0)\mu^{-r} < \infty.$$
A function f on $G_\mathbb{A}^0/G_k$ is said to be *slowly increasing* if $f \in C(G_\mathbb{A}^0/G_k, r)$ for some $r \in \mathbb{R}$.

Note that $C(G_\mathbb{A}^0/G_k, r_1) \subset C(G_\mathbb{A}^0/G_k, r_2)$ if $r_1 > r_2$ and $C(G_\mathbb{A}^0/G_k, r) \subset L^1(G_\mathbb{A}^0/G_k, dg^0)$ if $r > -2$.

Definition 4.3. Let $\Phi \in \mathcal{S}(V_\mathbb{A})$ and $\bar{g} \in \overline{G}_\mathbb{A}$. For any subset $L \subset V_k$, we define an incomplete theta series $\Theta_L(\Phi, \bar{g})$ by
$$\Theta_L(\Phi, \bar{g}) = \sum_{x \in L} \Phi(\bar{g}x).$$
If L is invariant under the action of a subgroup $H_k \subseteq G_k$, we often regard $\Theta_L(\Phi, \bar{g})$ as a function on $\bar{G}_\mathbb{A}/H_k$. For example, if $L = V_k^{ss}$, $\Theta_{V_k^{ss}}(\Phi, \bar{g})$ is a function on $\overline{G}_\mathbb{A}/G_k$.

Lemma 4.4. *For any $N \ge 1$,*
$$\Theta_{V_k^{ss}}(\Phi, \bar{g}) \ll \begin{cases} \lambda^{-2N-2}\mu^{-1} & \lambda \ge 1, \\ \lambda^{-2-m}\mu^{-1} & \lambda \le 1, \end{cases}$$

for $\bar{g} \in \mathbb{R}_+ \times \mathfrak{S}^0$.

Proof. By (1) of Lemma 4.1, we may assume $\bar{g} = \underline{\lambda}(1, a_2(\underline{\mu}^{-1}, \underline{\mu})), \mu \ll 1$. For $x = (x_1, x_2) \in V_k^{ss}$, we have $x_1 \neq 0$ and $x_2 \neq 0$. Note that the weight of $a(t_1, t_2) \in T_2$ with respect to each k-coordinate of x_1 and x_2 is t_1 and t_2, respectively. Hence, by (2) and (3) of Lemma 4.1, for any $N_1, N_2 \geq 1$,

$$\Theta_{V_k^{ss}}(\Phi, \bar{g}) \ll (\lambda^{-1}\mu)^{N_1}(\lambda^{-1}\mu^{-1})^{N_2} \max(1, \lambda^{-1}\mu)^{m-1} \max(1, \lambda^{-1}\mu^{-1})^{m-1}$$
$$\leq \lambda^{-N_1 - N_2} \mu^{N_1 - N_2} \max(1, \lambda^{2-2m}) \max(1, \mu^{m-1}) \max(1, \mu^{1-m})$$
$$\ll \lambda^{-N_1 - N_2} \max(1, \lambda^{2-2m}) \mu^{N_1 - N_2 + 1 - m}.$$

Note that $\max\{1, ab\} \leq \max\{1, a\} \cdot \max\{1, b\}$ for $a, b \geq 0$. For $\lambda \geq 1$, take $N_1 = N + m - 2, N_2 = N$. For $\lambda \leq 1$, take $N_1 = 3, N_2 = 1$. Then we have the proposition. $\square$

Now we define the global zeta function.

Definition 4.5. For $\Psi \in \mathcal{S}(V_{\mathbb{A}})$ and a complex variable s, we define

$$Z(\Phi, s) = \int_{\overline{G}_{\mathbb{A}}/G_k} \lambda^s \Theta_{V_k^{ss}}(\Phi, \bar{g}) \, d\bar{g},$$

$$Z_+(\Phi, s) = \int_{\substack{\overline{G}_{\mathbb{A}}/G_k \\ \lambda \geq 1}} \lambda^s \Theta_{V_k^{ss}}(\Phi, \bar{g}) \, d\bar{g}.$$

The integral $Z(\Phi, s)$ is called the *global zeta function*. By Lemma 4.4, the integral $Z(\Phi, s)$ converges absolutely and locally uniformly on a certain right half-plane and the integral $Z_+(\Phi, s)$ is an entire function. Since the global zeta function is well defined for V_k^{ss}, by Theorem (0.3.7) in [20] (which is due to Shintani), $Z(\Phi, s)$ can be continued meromorphically to the entire plane and satisfies a functional equation

$$Z(\Phi, s) = Z(\widehat{\Phi}, 2m - s).$$

The purpose of this section is to determine the pole structure and to describe the residues by means of certain distributions.

Remark 4.6. The above definition of the zeta function looks slightly different from the original definition in [15], but these are essentially the same. We briefly compare these functions. Let $\widetilde{G} = G/\widetilde{T}$. Recall that we put $\widetilde{T} = \ker(G \to \mathrm{GL}(V))$. Let $d\widetilde{g}$ be an invariant measure on $\widetilde{G}_{\mathbb{A}}$. The original definition of the global zeta function is as follows:

$$Z^*(\Phi, s) = \int_{\widetilde{G}_{\mathbb{A}}/\widetilde{G}_k} |\chi(\widetilde{g})|^s \Theta_{V_k^{ss}}(\Phi, \widetilde{g}) \, d\widetilde{g}.$$

Since $\widetilde{T} \cong \mathrm{GL}(1) \times \mathrm{GL}(1)$ is a split torus, the first Galois cohomology set $H^1(k', \widetilde{T})$ is trivial for any field k' containing k. This implies that the set of k'-rational point of $\widetilde{G}$ coincides with $G_{k'}/\widetilde{T}_{k'}$. Therefore $\widetilde{G}_{\mathbb{A}} = G_{\mathbb{A}}/\widetilde{T}_{\mathbb{A}}$ and $\widetilde{G}_{\mathbb{A}}/\widetilde{G}_k = G_{\mathbb{A}}/\widetilde{T}_{\mathbb{A}} G_k$. Let $\widetilde{T}_{\mathbb{A}}^0 = G_{\mathbb{A}}^0 \cap \widetilde{T}_{\mathbb{A}}$. Then we have

$$(\mathbb{R}_+ \times G_{\mathbb{A}}^0)/\widetilde{T}_{\mathbb{A}}^0 \cong G_{\mathbb{A}}/\widetilde{T}_{\mathbb{A}}$$

via the map which sends the class of $(\lambda, g_{11}, g_{12}, g_2)$ to class of $(g_{11}, g_{12}, \underline{\lambda}g_2)$. Moreover, this map is compatible with their actions on $V_{\mathbb{A}}$. If we identify $\overline{G}_{\mathbb{A}}/\widetilde{T}_{\mathbb{A}}^0$ with $\widetilde{G}_{\mathbb{A}}$ via the isomorphism, then we have $|\chi(\bar{g})| = \lambda^4$ for the D_4 case and $|\chi(\bar{g})| = \lambda^{12}$ for the E_6 case. Also the volume of $\widetilde{T}_{\mathbb{A}}^0/\widetilde{T}_k \cong (\mathbb{A}^0/k^\times)^2$ is finite. Hence $Z(\Phi, 4s)$ is a constant multiple of $Z^*(\Phi, s)$ for the D_4 case and $Z(\Phi, 12s)$ is a constant multiple of $Z^*(\Phi, s)$ for the E_6

case. (The constant depends on the choice of the measure.) Our choice of $Z(\Phi, s)$ is for the conventions of our global theory.

Let

$$\mathrm{M}\Phi(x) = \int_{K_2} \Phi(\kappa_2 x) \, d\kappa_2.$$

Then $Z(\Phi, s) = Z(\mathrm{M}\Phi, s)$ and $\mathrm{M}\Phi(x)$ is K_2-invariant. Therefore, we may assume the following for the rest of this section.

Assumption 4.7. The Schwartz-Bruhat function Φ is K_2-invariant.

§ 4.3. **The principal part.** For $x = (x_1, x_2)$ and $y = (y_1, y_2)$, we define

$$[x, y] = \mathcal{T}(x_1 y_2) + \mathcal{T}(x_2 y_1).$$

This is a non-degenerate bilinear form on V. For $\bar{g} = (\lambda, g_{11}, g_{12}, g_2) \in \mathbb{R}_+ \times G_{\mathbb{A}}^0$, we define

$$\bar{g}^\iota = (\lambda^{-1}, g_{12}^{-1}, g_{11}^{-1}, \nu^t g_2^{-1} \nu).$$

This is an involution and the above bilinear form satisfies

$$[\bar{g}x, \bar{g}^\iota y] = [x, y].$$

Recall that $\langle \, \rangle$ is a non-trivial additive character of $\mathbb{A}/k$. For $\Phi \in \mathcal{S}(V_{\mathbb{A}})$, we define its Fourier transform by

$$\widehat{\Phi}(x) = \int_{V_{\mathbb{A}}} \Phi(y) \langle [x, y] \rangle \, dy.$$

It is easy to see that the Fourier transform of $\Phi(\bar{g}\cdot)$ is $\lambda^{-8} \widehat{\Phi}(\bar{g}^\iota \cdot)$.

For $\lambda \in \mathbb{R}_+$, we define $\Phi_\lambda(x) = \Phi(\underline{\lambda} x)$.

Definition 4.8. For $\Phi \in \mathcal{S}(V_{\mathbb{A}}), s \in \mathbb{C}$ and $g^0 \in G_{\mathbb{A}}^0$, we define

$$J(\Phi, g^0) = \sum_{x \in S_k} \widehat{\Phi}((g^0)^\iota x) - \sum_{x \in S_k} \Phi(g^0 x),$$

$$I^0(\Phi) = \int_{G_{\mathbb{A}}^0/G_k} J(\Phi, g^0) \, dg^0,$$

$$I(\Phi, s) = \int_0^1 \lambda^s I^0(\Phi_\lambda) \, d^\times \lambda.$$

Then by the Poisson summation formula, we have the following.

Proposition 4.9. *We have*

$$Z(\Phi, s) = Z_+(\Phi, s) + Z_+(\widehat{\Phi}, 2m - s) + I(\Phi, s).$$

We study the last term for the rest of this section.

§ 4.4. **Stratification.** In this subsection, we consider a stratification of V_k. Let

$$Y_1 = \{x \in V \mid x_1 = 0\}, \quad Y_1^{\mathrm{ss}} = \{x \in Y_1 \mid x_2 \neq 0\}.$$

We define $S_1 = GY_1^{\mathrm{ss}}$. Let $P = G_1 \times B_2$.

Lemma 4.10. *We have*

(1) $V_k \setminus \{0\} = V_k^{\mathrm{ss}} \amalg S_{1k}$,

(2) $S_{1k} = G_k \times_{P_k} Y_{1k}^{\mathrm{ss}}$.

Proof. We consider (1). Let $x \in V_k \setminus \{0\}$ and $x \notin V_k^{\mathrm{ss}}$. Since either $x_1 \neq 0$ or $x_2 \neq 0$, there exists an element $g \in G_k$ such that the first coordinate of gx is 1. Replacing x by gx, we may assume that x is of the form $x = (1, -x_2)$, where $x_2 \in W_k$. Then $F_x(v)$ is the characteristic polynomial of x_2 and the condition $P(x) = 0$ is equivalent to that the characteristic polynomial of x_2 has a multiple root.

Let $L = k[x_2]$ be the subalgebra of D_k generated by x_2 over k. Since D_k has no zero divisor, $k[x_2]$ is a (commutative) integral domain which is finite over the field k. So it is a field. Then since the degree of extension $[L : k]$ divides $\dim_k D_k = n^2$, it is either 1 or n. Note that we are assuming $n = 2$ or 3. Assume $[L : k] = n$. Then $F_x(v)$ is a minimum polynomial of x_2 over k because the degree of $F_x(v)$ is n. Since any field extension of an algebraic number field is separable, we conclude that $F_x(v)$ does not have a multiple root. This is an contradiction and hence $[L : k] = 1$, which implies $x_2 \in k$. Therefore, there exists an element $g_2 \in G_{2k}$ such that $g_2 x \in Y_{1k}^{\mathrm{ss}}$. This proves (1).

It is easy to see that $P_{1k} Y_{1k}^{\mathrm{ss}} = Y_{1k}^{\mathrm{ss}}$ and that if $x \in Y_{1k}^{\mathrm{ss}}, g \in G_k$ and $gx \in Y_{1k}^{\mathrm{ss}}$ then $g \in P_k$. This proves (2). $\qquad \square$

§ 4.5. **The smoothed Eisenstein series.** To compute $I^0(\Phi)$, it seems natural to divide the index set S_k of the summation into its G_k-orbits and perform integration separately. However, we can not put this into practice because the corresponding integrals diverge. This is the main difficulty when one calculates the global zeta functions of the prehomogeneous vector spaces. To surmount this problem Shintani [16] introduced the smoothed Eisenstein series of GL(2). He used this series to determine the principal parts of the global zeta functions for the space of binary cubic forms and the space of binary quadratic forms. Later A. Yukie [20] generalized the theory of Eisenstein series to the products of GL(n)'s, and applied it to determine the principal parts of the global zeta functions in some cases. In this subsection, we essentially repeat the argument of Shintani and Yukie in our settings.

We express the Iwasawa decomposition of $g_2 \in G_{2\mathbb{A}}^0$ as

$$g_2 = \kappa_2(g_2) a_2(t_{21}(g_2), t_{22}(g_2)) n_2(u(g_2)).$$

Let $s \in \mathbb{C}$. The Eisenstein series of $G_{2\mathbb{A}}^0$ for B_2 is defined as

$$E(g_2, s) = \sum_{\gamma \in G_{2k}/B_{2k}} |t_{21}(g_2 \gamma)|^{s+1}$$

It is well known that the summation defining $E(g_2, s)$ converges absolutely and locally uniformly in certain right half-plane and can be continued meromorphically to the whole complex plane. For analytic properties of $E(g_2, s)$, see [18], [20].

Let $\psi(s)$ be an entire function of s such that

$$\sup_{c_1 < \Re(s) < c_2} (1 + |s|^N) |\psi(s)| < \infty$$

for all $c_1 < c_2, N > 0$. Moreover, we assume $\psi(1) \neq 0$. For a complex variable w, we define

$$\Lambda_\psi(w; s) = \frac{\psi(s)}{w - s}.$$

Where there is no confusion, we drop ψ and use the notation $\Lambda(w; s)$ instead.

Definition 4.11. For a complex variable w, we define

$$\mathcal{E}(g^0, w, \psi) = \frac{1}{2\pi\sqrt{-1}} \int_{\Re(s)=r_1} E(g_2, s)\Lambda_\psi(w; s)\, ds.$$

for some $r_1 > 1$.

Note that the above definition does not depend on the choice of r_1. The function $\mathcal{E}(g^0, w, \psi)$ is called the smoothed Eisenstein series. When there is no confusion, we drop ψ and use the notation $\mathcal{E}(g^0, w)$ instead.

The following proposition is known as Shintani's lemma.

Proposition 4.12. (1) *The function $\mathcal{E}(g^0, w)$ is holomorphic for $\Re(w) > 0$ except for a simple pole at $w = 1$ with the residue $\varrho\psi(1)$.*

(2) *Let $f \in C(G^0_\mathbb{A}/G_k, r)$ for some $r > -2$. Then the integral*

$$\int_{G^0_\mathbb{A}/G_k} f(g^0)\mathcal{E}(g^0, w)\, dg^0$$

becomes a holomorphic function for $\Re(w) > 1 - \epsilon$ for a constant $\epsilon > 0$ except possibly for a simple pole at $w = 1$ with residue

$$\varrho\psi(1) \int_{G^0_\mathbb{A}/G_k} f(g^0)\, dg^0.$$

(3) *For a slowly increasing function $f(g^0)$ on $G^0_\mathbb{A}/G_k$, the integral*

$$\int_{G^0_\mathbb{A}/G_k} f(g^0)\mathcal{E}(g^0, w)\, dg^0$$

becomes a holomorphic function on a certain right half-plane.

(4) *We have*

$$\int_{G^0_\mathbb{A}/G_k} \mathcal{E}(g^0, w)\, dg^0 = \tau(G_1)\Lambda(w; 1).$$

The above proposition was first proved for GL(2) by Shintani [16, pp. 172, 173, 177]. The adelic proof is given in [18, pp. 527, 528]. In our case, we included the first factor $G^0_{1\mathbb{A}}$ in the statement instead of just considering GL(2), but exactly the same proof works because $G^0_{1\mathbb{A}}/G_{1k}$ is compact. For the convenience of the reader, we give a sketch of the proof here. To prove Proposition 4.12, the following lemma on the Eisenstein series for $G_2 = \mathrm{GL}(2)$ is crucial. For the proof, see [16], [18], or [20].

Lemma 4.13. (1) *Let $E_{N_2}(g_2, s)$ be the constant term of $E(g_2, w)$ with respect to N_2 i.e.*

$$E_{N_2}(g_2, s) = \int_{N_{2\mathbb{A}}/N_{2k}} E(g_2 n_2(u), s)\, du.$$

The constant term has the following explicit formula:

$$E_{N_2}(g_2, s) = \mu^{-s-1} + \mu^{s-1}\phi(s).$$

(2) *Let $\widetilde{E}(g_2, s) = E(g_2, s) - E_{N_2}(g_2, s)$ be the non-constant term. Then $\widetilde{E}(g_2, s)$ is holomorphic for $\Re(s) > 0$. Moreover, for any s in this region and $l > 1$,*

$$|\widetilde{E}(g_2, s)| \ll \mu^{2l-1}.$$

(3) *We have*

$$\int_{G_{2\mathbb{A}}^0/G_{2k}} \mathcal{E}(g_2, w)\, dg_2 = \Lambda(w, 1).$$

Sketch of the proof of Proposition 4.12. Let

$$E'(g_2, s) = E(g_2, s) - \mu^{s-1}\phi(s) = \mu^{-s-1} + \widetilde{E}(g_2, s).$$

This function is holomorphic for $\Re(s) > 0$. Recall that $\rho = \operatorname{Res}_{s=1}\phi(s)$.

Let ϵ be a small positive number. By moving the contour, we have

$$\frac{1}{2\pi\sqrt{-1}}\int_{\Re(s)=r_1} \mu^{s-1}\phi(s)\frac{\psi(s)}{w-s}\, ds = \frac{\rho\psi(1)}{w-1} + \frac{1}{2\pi\sqrt{-1}}\int_{\Re(s)=\epsilon} \mu^{s-1}\phi(s)\frac{\psi(s)}{w-s}\, ds,$$

$$\frac{1}{2\pi\sqrt{-1}}\int_{\Re(s)=r_1} E'(g_2, s)\frac{\psi(s)}{w-s}\, ds = \frac{1}{2\pi\sqrt{-1}}\int_{\Re(s)=\epsilon} E'(g_2, s)\frac{\psi(s)}{w-s}\, ds,$$

and each of the integrals in the right hand sides is holomorphic for $\Re(w) > \epsilon$. This proves (1).

We consider (2). Since $C(G_{\mathbb{A}}^0/G_k, r_1) \subset C(G_{\mathbb{A}}^0/G_k, r_2)$ for $r_1 > r_2$, we may assume $-2 < r < 0$. By moving the contour, we have

$$\frac{1}{2\pi\sqrt{-1}}\int_{\Re(s)=r_1} \left(\int_{G_{\mathbb{A}}^0/G_k} f(g^0)\mu^{s-1}\, dg^0\right)\phi(s)\frac{\psi(s)}{w-s}\, ds$$

$$= \frac{\rho\psi(1)}{w-1}\int_{G_{\mathbb{A}}^0/G_k} f(g^0)\, dg^0$$

$$+ \frac{1}{2\pi\sqrt{-1}}\int_{\Re(s)=-r/2} \left(\int_{G_{\mathbb{A}}^0/G_k} f(g^0)\mu^{s-1}\, dg^0\right)\phi(s)\frac{\psi(s)}{w-s}\, ds.$$

The second term in the right hand side is holomorphic if $\Re(w) > -r/2$ because for $\Re(s) = -r/2$, we have $f(g^0)\mu^{s-1} \in C(G_{\mathbb{A}}^0/G_k, (r-2)/2) \subset L^1(G_{\mathbb{A}}^0/G_k, dg^0)$. A similar argument shows that

$$\frac{1}{2\pi\sqrt{-1}}\int_{\Re(s)=r_1} \left(\int_{G_{\mathbb{A}}^0/G_k} f(g^0)E'(g_2, s)\, dg^0\right)\frac{\psi(s)}{w-s}\, ds$$

is holomorphic for $\Re(w) > 0$. This proves (2).

To see (3), we may assume $f(g^0) \in C(G_{\mathbb{A}}^0/G_k, r)$ for some $r \leq -2$. Then $f(g^0)\mu^{s-1} \in C(G_{\mathbb{A}}^0/G_k, -1)$ for $\Re(s) \geq -r$, $f(g^0)\mu^{-s-1} \in C(G_{\mathbb{A}}^0/G_k, -1)$ for $\Re(s) \leq r$, and $f(g^0)\widetilde{E}(g_2, s) \in C(G_{\mathbb{A}}^0/G_k, -1)$ for $\Re(s) > \epsilon$ where $\epsilon > 0$ is a small number. Hence by moving the contour for each of these terms, we obtain (3). Now (4) immediately follows from Lemma 4.13 (3), and these finish the proof of Proposition 4.12. $\square$

Let $\mathcal{E}_{N_2}(g^0, w)$ be the constant term of $\mathcal{E}(g^0, w)$ with respect to N_2 i.e.

$$\mathcal{E}_{N_2}(g^0, w) = \int_{N_{2\mathbb{A}}/N_{2k}} \mathcal{E}(g^0 n(u), w)\, du.$$

By Lemma 4.13 (1), we have

$$\mathcal{E}_{N_2}(g^0, w) = \frac{1}{2\pi\sqrt{-1}}\int_{\Re(s)=r_1} (\mu^{-s-1} + \mu^{s-1}\phi(s))\Lambda(w; s)\, ds.$$

Definition 4.14. Let $f(w), g(w)$ be holomorphic functions of $w \in \mathbb{C}$ in some right half-plane. We use the notation $f(w) \sim g(w)$ if $f(w) - g(w)$ can be continued meromorphically to $\{w \mid \Re(w) > 1 - \epsilon\}$ for some $\epsilon > 0$ and is holomorphic at $w = 1$.

We define

$$I^0(\Phi, w) = \int_{G_\mathbb{A}^0/G_k} J(\Phi, g^0)\mathcal{E}(g^0, w)\, dg^0.$$

By Lemma 4.4, $J(\Phi, g) \in C(G_\mathbb{A}^0/G_k, -1)$. Hence, by Proposition 4.12 (2), we have the following.

Proposition 4.15. *We have*

$$I^0(\Phi, w) \sim \varrho\Lambda(w; 1)I^0(\Phi).$$

Definition 4.16. For a complex variable w, we define

$$\Xi_1(\Phi, w) = \int_{G_\mathbb{A}^0/G_k} \Theta_{S_{1k}}(\Phi, g^0)\mathcal{E}(g^0, w)\, dg^0,$$

$$\Xi_\#(\Phi, w) = \Phi(0)\int_{G_\mathbb{A}^0/G_k} \mathcal{E}(g^0, w)\, dg^0.$$

Since $\Theta_{S_k'}(\Phi, g^0)$ is a slowly increasing function, by Proposition 4.12 (3), the integral $\Xi_1(\Phi, w)$ converges absolutely for sufficiently large $\Re(w)$. It is proved in [20] that $E(g_2, s) = E({}^tg_2^{-1}, s)$ for $g_2 \in G_{2\mathbb{A}}^0$. Hence, $\mathcal{E}(g^0, w) = \mathcal{E}((g^0)^\iota, w)$ for $g^0 \in G_\mathbb{A}^0$. Therefore, by Lemma 4.10, we have the following.

Proposition 4.17. *We have*

$$I^0(\Phi, w) = \Xi_1(\widehat{\Phi}, w) + \Xi_\#(\widehat{\Phi}, w) - \Xi_1(\Phi, w) - \Xi_\#(\Phi, w).$$

For $\Xi_\#(\Phi, w)$, Proposition 4.12 immediately leads to the following.

Proposition 4.18. *We have*

$$\Xi_\#(\Phi, w) = \Phi(0)\tau(G_1)\mathfrak{V}_2\varrho\Lambda(w, \rho).$$

We study $\Xi_1(\Phi, w)$ in §4.7.

§ 4.6. The zeta function associated with the space of division algebra. Since the prehomogeneous vector space (G_1, W) of (single) division algebra appears in the induction process, we have to know the principal part of the zeta function for this case. This function is essentially the same as that of Godement-Jacquet [7]. In this subsection we review the principal part of the zeta function in this case.

We put $P_1(x) = \mathcal{N}(x)$ for $x \in W$ and $W^{\mathrm{ss}} = \{x \in W \mid P_1(x) \neq 0\}$. Note that $W_k^{\mathrm{ss}} = \{x \in W_k \mid x \neq 0\}$. The group $\mathbb{R}_+ \times G_{1\mathbb{A}}^0$ acts on $W_\mathbb{A}$ by assuming that $\lambda \in \mathbb{R}_+$ acts by multiplication by $\underline{\lambda}$. For any subset $L \subset W_k$, we define $\Theta_L(\Psi, \underline{\lambda}g_1)$ in the obvious manner.

Definition 4.19. For $\Psi \in \mathcal{S}(W_\mathbb{A})$ and $s \in \mathbb{C}$,

$$Z_W(\Psi, s) = \int_{\mathbb{R}_+ \times G_{1\mathbb{A}}^0/G_{1k}} \lambda^s \Theta_{W_k^{\mathrm{ss}}}(\Psi, \underline{\lambda}g_1)\, d^\times\lambda dg_1^0,$$

$$Z_{W+}(\Psi, s) = \int_{\substack{\mathbb{R}_+ \times G_{1\mathbb{A}}^0/G_{1k} \\ \lambda \geq 1}} \lambda^s \Theta_{W_k^{\mathrm{ss}}}(\Psi, \underline{\lambda}g_1)\, d^\times\lambda dg_1^0.$$

The following lemma is a direct consequence of Lemma 4.1.

Lemma 4.20. *The integral defining $Z_W(\Phi, s)$ converges absolutely and locally uniformly in the region $\Re(s) > m$, and the integral defining $Z_{W+}(\Phi, s)$ is an entire function.*

For $x, y \in W$, we put
$$[x, y]_W = \mathfrak{T}(xy)$$
This defines a non-degenerate bilinear form on W. We note that this bilinear form satisfies $[g_1 x, g_1^\iota y]_W = [x, y]_W$ where we put $(g_{11}, g_{12})^\iota = (g_{12}^{-1}, g_{11}^{-1})$.

We define the Fourier transform on $\mathcal{S}(W_\mathbb{A})$ by
$$\Psi^*(x) = \int_{W_\mathbb{A}} \Psi(y) \langle [x, y]_W \rangle \, dy.$$

Then by the Poisson summation formula, we have
$$\Theta_{W_k^{ss}}(\Psi, \underline{\lambda} g_1) = \lambda^{-m} \Theta_{W_k^{ss}}(\Psi^*, \underline{\lambda}^{-1}(g_1)^\iota) + \lambda^{-m} \Psi^*(0) - \Psi(0).$$

Applying the above equation, we obtain the following principal part formula for this zeta function.

Proposition 4.21. *We have*
$$Z_W(\Psi, s) = Z_{W+}(\Psi, s) + Z_{W+}(\Psi^*, m - s) + \tau(G_1)\left(\frac{\Psi^*(0)}{s - m} - \frac{\Psi(0)}{s}\right),$$

where $Z_{W+}(\Psi, s)$ and $Z_{W+}(\Psi^*, m - s)$ are entire functions.

§ 4.7. **Contribution from unstable strata.** In this subsection, we express the residue of $\Xi_1(\Phi, w)$ in terms of Z_W defined in the previous subsection. We identify Y_1 (see §4.4) with the space W of single division algebras in §4.6.

Definition 4.22. For $\Phi \in \mathcal{S}(V_\mathbb{A})$, we define a Schwartz-Bruhat function $\mathcal{R}_W \Phi$ on $W_\mathbb{A}$ by restricting to $Y_{1\mathbb{A}}$.

Proposition 4.23. *By changing ψ if necessary, we have*
$$\Xi_1(\Phi, w) \sim \rho \Lambda(w; 1) Z_W(\mathcal{R}_W \Phi, 2).$$

Proof. We have
$$\Xi_1(\Phi, w) = \int_{G_\mathbb{A}^0 / G_k} \Theta_{S_{1k}}(\Phi, g^0) \mathcal{E}(g^0, w) \, dg^0$$
$$= \int_{G_\mathbb{A}^0 / P_k} \Theta_{Y_{1k}^{ss}}(\Phi, g^0) \mathcal{E}(g^0, w) \, dg^0$$
$$= \int_{P_\mathbb{A}^0 / P_k} \Theta_{Y_{1k}^{ss}}(\Phi, p^0) \mathcal{E}(p^0, w) \, dg^0$$
$$= \int_{G_{1\mathbb{A}}^0 / G_{1k} \times B_{2\mathbb{A}}^0 / B_{2k}} \Theta_{Y_{1k}^{ss}}(\Phi, (g_1, b_2)) \mathcal{E}(b_2, w) \, dg_1 db_2$$
$$= \int_{G_{1\mathbb{A}}^0 / G_{1k} \times T_{2\mathbb{A}}^0 / T_{2k}} \mu^2 \Theta_{Y_{1k}^{ss}}(\Phi, (g_1, t_2)) \mathcal{E}_{N_2}(t_2, w) \, dg_1 d^\times t_2.$$

The last step is because N_2 acts on Y_1 trivially. By changing g_{11} to $g_{11} t_{22}^{-1}$, we have
$$\Xi_1(\Phi, w) = \int_{\mathbb{R}_+ \times G_{1\mathbb{A}}^0 / G_{1k}} \mu^2 \Theta_{W_k^{ss}}(\mathcal{R}_W \Phi, (\mu, g_1)) \mathcal{E}_{N_2}(a(\underline{\mu}^{-1}, \underline{\mu}), w) \, d^\times \mu dg_1.$$

By the definition of $Z_W(\Psi, s)$, we have

$$\int_{\mathbb{R}_+ \times G_{1\mathbb{A}}^0/G_{1k}} \mu^{\mp s+1} \Theta_{W_k^{ss}}(\mathcal{R}_W \Phi, (\mu, g_1)) \, d^\times \mu dg_1 = Z_W(\mathcal{R}_W \Phi, \mp s + 1)$$

for $\Re(s) < -3, \Re(s) > 3$, respectively. Since

$$\frac{1}{2\pi\sqrt{-1}} \int_{\Re(s)=r_2<-3} Z_W(\mathcal{R}_W \Phi, -s+1) \Lambda(w; s) \, ds \sim 0,$$

we have

$$\Xi_1(\Phi, w) \sim \frac{1}{2\pi\sqrt{-1}} \int_{\Re(s)=r_3>3} Z_W(\mathcal{R}_W \Phi, s+1) \phi(s) \Lambda(w; s) \, ds.$$

By Proposition 4.21, $Z_W(\mathcal{R}_W \Phi, s+1)$ has a simple pole at $s = 3$ and is holomorphic for $\Re(s) > 0$ except for that point. If we consider $(s-3)\psi(s)$ instead of $\psi(s)$, this function still satisfies the property we have assumed. Namely,

$$\sup_{c_1 < \Re(s) < c_2} (1 + |s|^N)|(s-3)\psi(s)| < \infty$$

for all $c_1 < c_2, N > 0$ and $(s-3)\psi(s)|_{s=1} \neq 0$. Therefore, by changing $\psi(s)$ to $(s-3)\psi(s)$, we may assume that $Z_W(\mathcal{R}_W \Phi, s+1)\Lambda(w; s)$ is holomorphic for $\Re(s) > 0$. (This is the passing principle (3.6.1) of [20].) Hence,

$$\Xi_1(\Phi, w) \sim \frac{1}{2\pi\sqrt{-1}} \int_{\Re(s)=1/2} Z_W(\mathcal{R}_W \Phi, s+1) \phi(s) \Lambda(w; s) \, ds$$
$$+ \varrho\Lambda(w; 1) Z_W(\mathcal{R}_W \Phi, 2)$$
$$\sim \varrho\Lambda(w; 1) Z_W(\mathcal{R}_W \Phi, 2)$$

This proves the proposition. $\qquad\square$

§4.8. The principal part formula.

Theorem 4.24. *Suppose that $\Phi = M\Phi$. Then*

$$Z(\Phi, s) = Z_+(\Phi, s) + Z_+(\widehat{\Phi}, 2m - s)$$
$$+ \tau(G_1)\mathfrak{V}_2 \left(\frac{\widehat{\Phi}(0)}{s - 2m} - \frac{\Phi(0)}{s} \right) + \frac{Z_W(\mathcal{R}_W \widehat{\Phi}, 2)}{s - (2m-2)} - \frac{Z_W(\mathcal{R}_W \Phi, 2)}{s - 2},$$

where the first two terms in the right hand side are entire functions.

Proof. By Proposition 4.17, 4.18 and 4.23,

$$I^0(\Phi, w) = \rho\Lambda(w; 1) \left(\tau(G_1)\mathfrak{V}_2(\widehat{\Phi}(0) - \Phi(0)) + Z_W(\mathcal{R}_W \Phi, 2) - Z_W(\mathcal{R}_W \widehat{\Phi}, 2) \right)$$

for a suitable choice of $\psi(s)$. Hence, together with Proposition 4.15, we obtain

$$I^0(\Phi) = \tau(G_1)\mathfrak{V}_2(\widehat{\Phi}(0) - \Phi(0)) + Z_W(\mathcal{R}_W \Phi, 2) - Z_W(\mathcal{R}_W \widehat{\Phi}, 2).$$

Recall that $I(\Phi, s) = \int_0^1 I^0(\Phi_\lambda) d^\times \lambda$ where $\Phi_\lambda(x) = \Phi(\underline{\lambda}x)$. It is easy to see that

$$\Phi_\lambda(0) = \Phi(0), \quad \widehat{\Phi_\lambda}(0) = \lambda^{-2m}\widehat{\Phi}(0).$$

Since

$$Z_W(\mathcal{R}_W \Phi_\lambda, s) = \lambda^{-s} Z_W(\mathcal{R}_W \Phi, s),$$
$$Z_W(\mathcal{R}_W \widehat{\Phi_\lambda}, s) = \lambda^{2m-s} Z_W(\mathcal{R}_W \widehat{\Phi}, s),$$

we get

$$Z_W(\mathcal{R}_W\Phi_\lambda, 2) = \lambda^{-2}Z_W(\mathcal{R}_W\Phi, 2),$$
$$Z_W(\mathcal{R}_W\widehat{\Phi_\lambda}, 2) = \lambda^{2m-2}Z_W(\mathcal{R}_W\widehat{\Phi}, 2).$$

Then the theorem follows by integrating $\lambda^s I^0(\Phi_\lambda)$ over $s \in (0,1]$. $\qquad\square$

Theorem 1.2 in the introduction immediately follows from the above theorem.

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